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MOMENTS OF A RANDOM VARIABLE ARISING FROM LAPLACIAN RANDOM VARIABLE

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ABSTRACT. Let X be the Laplacian random variable with parameters $(a,b)=(0,1)$, and let $(X_j)_{j\geq 1}$ be a sequence of mutually independent copies of X. In this note, we explicitly determine the moments of the random variable $\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}$ in terms of the Bernoulli and Euler numbers.

1. INTRODUCTION

The Bernoulli numbers B_n and the Euler numbers E_n are respectively defined by

$$(1)\qquad \frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\quad \frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\quad (\text{see } [1-4]).$$

The first few terms of B_n are given by:

$$(2)\quad B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42}, B_8=-\frac{1}{30}, B_{10}=\frac{5}{66},$$
$$B_{12}=-\frac{691}{2730}, B_{14}=\frac{7}{6}, B_{16}=-\frac{3617}{510}, B_{18}=\frac{43867}{798}, B_{20}=-\frac{174611}{330},\dots;$$
$$B_{2k+1}=0,\ (k\geq 1).$$

The first few terms of E_n are given by:

$$(3)\quad E_0=1, E_1=-\frac{1}{2}, E_3=\frac{1}{4}, E_5=-\frac{1}{2}, E_7=\frac{17}{8}, E_9=-\frac{31}{2}, E_{11}=\frac{691}{4},$$
$$E_{13}=-\frac{5461}{2}, E_{15}=\frac{929569}{16}, E_{17}=-\frac{3202291}{2}, E_{19}=\frac{221930581}{4},\dots;$$
$$E_{2k}=0,\ (k\geq 1).$$

A random variable X is the Laplacian random variable with parameters a and $b(>0)$, which is denoted by $X\sim L(a,b)$, if its probability density function is given by

$$(4)\qquad f(x)=\frac{1}{2b}e^{-\frac{|x-a|}{b}},\ x\in(-\infty,\infty),\quad (\text{see } [5,7]),$$

where a is the local parameter and $b(>0)$ is the scale parameter.

The Euler's product expansion for the sine function is the identity

$$(5)\qquad \frac{\sin\pi x}{\pi x}=\prod_{j=1}^{\infty}\left(1-\frac{x^2}{j^2}\right),\quad \big(x\in(-\infty,\infty)\big),\quad (\text{see } [2]).$$

This identity was used by Euler in 1735 to give a solution of the Basel problem.

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Let $X \sim L(0,1)$, and let $(X_j)_{j\geq 1}$ be a sequence of mutually independent copies of X. In this note, we determine the moments of the random variable $Y=\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}$. Indeed, we show that $E[Y^{2n}]=(-1)^n\big(\frac{2n}{2^{2n}}E_{2n-1}+B_{2n}\big)$, $(n\in\mathbb{N})$, and that all odd moments of Y vanish (see Theorem 2.1).

2. Moments of a random variable arising from Laplacian random variable

For $X\sim L(0,1)$, let us assume that $(X_j)_{j\geq 1}$ is a sequence of mutually independent copies of the random variable X. From (4), we note that

$$\begin{aligned}
E\left[e^{\frac{X_k}{2\pi k}t}\right] &= \frac{1}{2}\int_{-\infty}^{\infty}e^{\frac{x}{2\pi k}t}e^{-|x|}dx \\
&= \frac{1}{2}\left[\int_{-\infty}^{0}e^{x\left(1+\frac{t}{2\pi k}\right)}dx+\int_{0}^{\infty}e^{-\left(1-\frac{t}{2\pi k}\right)x}dx\right] \\
&= \frac{1}{2}\left[\frac{1}{1+\frac{1}{2\pi k}t}+\frac{1}{1-\frac{t}{2\pi k}}\right]=\frac{1}{1-\left(\frac{t}{2\pi k}\right)^2},
\end{aligned} \tag{6}$$

where k is a positive integer and $-2\pi<t<2\pi$.

Thus, by (6), we get

$$\begin{aligned}
\prod_{k=1}^{\infty}\left(\frac{1}{1-\left(\frac{t}{2\pi k}\right)^2}\right) &= \prod_{k=1}^{\infty}E\left[e^{\frac{X_k}{2\pi k}t}\right]=E\left[\prod_{k=1}^{\infty}e^{\frac{X_k}{2\pi k}t}\right] \\
&= E\left[e^{\sum_{k=1}^{\infty}\frac{X_k}{2\pi k}t}\right].
\end{aligned} \tag{7}$$

On the other hand, by (5), we get

$$\begin{aligned}
\prod_{k=1}^{\infty}\left(\frac{1}{1-\left(\frac{t}{2\pi k}\right)^2}\right) &= \frac{\frac{t}{2}}{\sin\frac{t}{2}}=\frac{\frac{t}{2}}{\frac{e^{\frac{it}{2}}-e^{-\frac{it}{2}}}{2i}}=i\frac{t}{e^{\frac{it}{2}}-e^{-\frac{it}{2}}} \\
&= it\left(\frac{e^{\frac{it}{2}}-1+1}{e^{it}-1}\right)=\frac{it}{2}\left(\frac{2}{e^{\frac{it}{2}}+1}\right)+\frac{it}{e^{it}-1} \\
&= \frac{it}{2}\sum_{n=0}^{\infty}E_n\frac{\left(\frac{it}{2}\right)^n}{n!}+\sum_{n=0}^{\infty}B_n\frac{(it)^n}{n!} \\
&= \frac{it}{2}+\frac{it}{2}\sum_{n=1}^{\infty}E_{2n-1}\frac{\left(\frac{it}{2}\right)^{2n-1}}{(2n-1)!}+1-\frac{it}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(it)^{2n} \\
&= 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)!}E_{2n-1}\left(\frac{t}{2}\right)^{2n}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}B_{2n}t^{2n} \\
&= 1+\sum_{n=1}^{\infty}(-1)^n\left(\frac{2n}{2^{2n}}E_{2n-1}+B_{2n}\right)\frac{t^{2n}}{(2n)!}.
\end{aligned} \tag{8}$$

By (7) and (8), we get

$$\begin{aligned}
1+\sum_{n=1}^{\infty}(-1)^n\left(\frac{2n}{4^n}E_{2n-1}+B_{2n}\right)\frac{t^{2n}}{(2n)!} &= \prod_{k=1}^{\infty}\left(\frac{1}{1-\left(\frac{t}{2k\pi}\right)^2}\right)=E\left[e^{\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}t}\right] \\
&= \sum_{n=0}^{\infty}E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^n\right]\frac{t^n}{n!}.
\end{aligned} \tag{9}$$

Therefore, by comparing the coefficients on both sides of (9), we obtain the following theorem.

Theorem 2.1. *For $X \sim L(0,1)$, let $(X_j)_{j\geq 1}$ be a sequence of mutually independent copies of the random variable X. Then we have*

$$E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{2n}\right] = (-1)^n\left(\frac{2n}{2^{2n}}E_{2n-1}+B_{2n}\right),$$

and

$$E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{2n-1}\right] = 0, \quad (n\in\mathbb{N}).$$

Remark 2.2. *As is known, the Bernoulli and Euler numbers are related by:*

$$E_n = -\frac{2(2^{n+1}-1)}{n+1}B_{n+1}, \quad (n\geq 0). \tag{10}$$

For example, this follows from the equation (14) of [6]. Thus, from Theorem 2.1 and (10)*, we have the following alternative expression:*

$$\begin{aligned} E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{2n}\right] &= (-1)^{n-1}\left(1-\frac{1}{2^{2n-1}}\right)B_{2n} \\ &= \left(1-\frac{1}{2^{2n-1}}\right)|B_{2n}|, \quad (n\in\mathbb{N}). \end{aligned} \tag{11}$$

Thus we have

$$E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{2n}\right] \sim |B_{2n}|, \text{ as } n\to\infty, \quad E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{2n+1}\right] = |B_{2n+1}|, \quad (n\in\mathbb{N}).$$

Finally, we illustrate Theorem 2.1 by using (11).

$$E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{2}\right] = \frac{1}{12}, \quad E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{4}\right] = \frac{7}{240},$$

$$E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{6}\right] = \frac{31}{1344}, \quad E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{8}\right] = \frac{127}{3840},$$

$$E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{10}\right] = \frac{2555}{33792}, \quad E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{12}\right] = \frac{1414477}{5591040},$$

$$E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{14}\right] = \frac{57337}{49152}, \quad E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{16}\right] = \frac{118518239}{16711680},$$

$$E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{18}\right] = \frac{5749691557}{104595456}, \quad E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}\right)^{20}\right] = \frac{91546277357}{173015040}.$$

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